SIMPLE SINE APPROXIMATIONS

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The exact sine and cosine functions are infinitely long polynomials – obviously impossible to calculate. Approximations are needed. Here's the result of some work we did.

A recent design project needed to create a circular motion using X and Y positioners driven by a microcontroller. X is proportional to cosine, Y to sine (Figure 1). The positioning calculations had to be relatively quick and fit within the overall control program. Only modest accuracy was required. We pursued mathematical approximations. The exact, infinitely long, equations are:

$$\sin(\Theta) = \Theta - \frac{\Theta^3}{3!} + \frac{\Theta^5}{5!} - \frac{\Theta^7}{7!} + \cdots,$$

$$\cos(\Theta) = 1 - \frac{\Theta^2}{2!} + \frac{\Theta^4}{4!} - \frac{\Theta^6}{6!} + \cdots,$$

where Θ is the angle in radians (not degrees: 1 radian = (180/π) degrees (57.296 degrees) and

! is the factorial symbol: 2! = (2 x 1) = 2,
3! = (3 x 2 x 1) = 6, 4! = (4 x 3 x 2 x 1) = 24, etc.

For small angles the terms farther out get smaller and smaller and usually can be ignored. The general approach is, create an approximation for angles between plus and minus 45 degrees, then use simple sine/cosine relations to apply it to larger angles. We'll discuss that in a while.

<u>Approximation 1 – Excellent</u>. Drop all terms higher than Θ^6. Remember, radians, not degrees.

$$\sin(\Theta) = \Theta - \frac{\Theta^3}{6} + \frac{\Theta^5}{120}$$

$$\cos(\Theta) = 1 - \frac{\Theta^2}{2} + \frac{\Theta^4}{24} - \frac{\Theta^6}{720}$$

For angles up to $\pm$45 degrees sine is accurate to $\pm$0.0004, cosine to $\pm$0.00004.

<u>Approximation 2 – Good</u>. Use only two terms. We experimentally modified the denominators for best accuracy.

$$\sin(\Theta) = \Theta - \frac{\Theta^3}{6.16}$$

$$\cos(\theta) = 1 - \frac{\theta^2}{2.085}$$

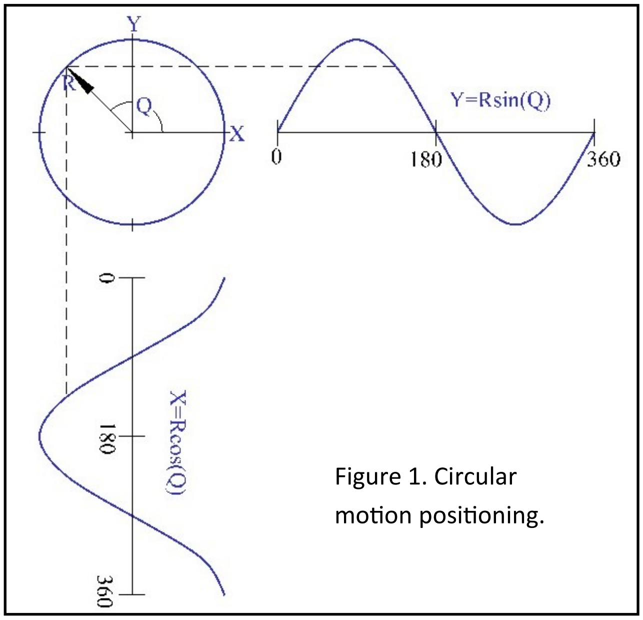

Figure 1. Circular motion positioning.

For angles up to $\pm$45 degrees, sine is accurate to $\pm$0.004, cosine to $\pm$0.003.

What do you do for angles greater than 45 degrees? You relate them to angles that aren't. Figure 2 graphs the sine and cosine functions, divided into 90 degree wide sections.

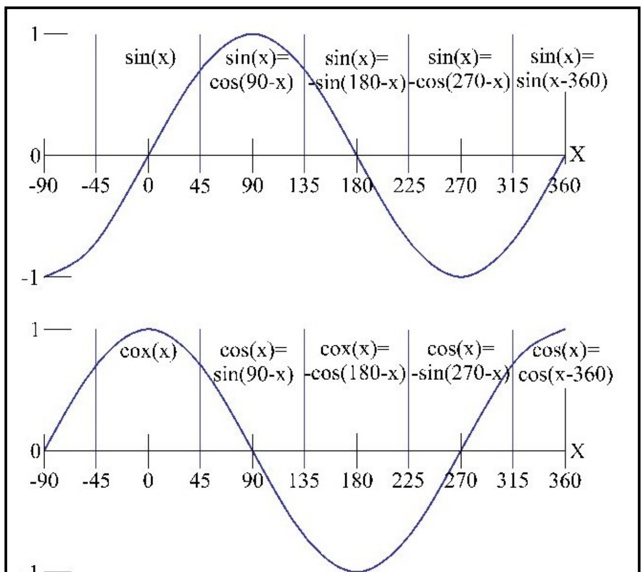

Figure 2. Sine/cosine relationships.

If you look at the graphs you can see that each section can be related back to ±45 degrees.

For sine:

Between 45 & 135 degrees, sin(Θ) = cos(90-Θ).

Between 135 & 225 degrees, sin(Θ) = -sin(180-Θ). Note the minus sign.

Between 225 & 315 degrees, sin(Θ) = -cos(270-Θ). Again, note the minus sign.

Above 315 degrees, sin(Θ) = sin(Θ-360).

For cosine:

Between 45 & 135 degrees, cos(Θ) = sin(90-Θ).

Between 135 & 225 degrees, cos(Θ) = -cos(180-Θ). Note the minus sign.

Between 225 & 315 degrees, cos(Θ) = -sin(270-Θ). Again, note the minus sign.

Above 315 degrees, cos(Θ) = cos(Θ-360).

These conversions are easy to program. If your angle is above 360 or below zero, subtract or add 360 enough times to bring it back to between 0 and 360 degrees, Then, apply these rules.

Approximation 3 - Simpler, fairly good.

Our positioning application did not require high accuracy and we wanted to minimize computation time. We decided to approximate the sine by straight line segments as shown in Figure 3. This requires only addition and multiplication, no squares or higher powers. After some computerized trials we set four line segments as follows. (The angle Θ is in degrees, not radians):

Between 0 & 30 degrees:
sin(Θ) = 0.01667(Θ).

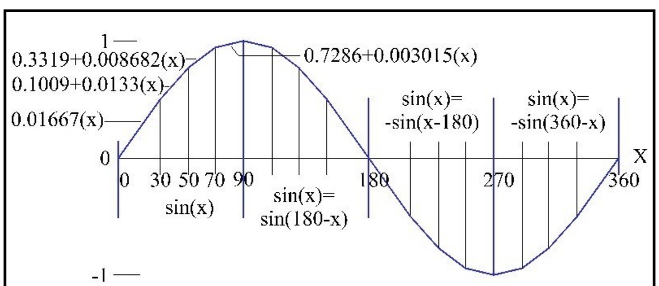

Figure 3. Sine approximation using line segments.

Between 30 & 50 degrees:
sin(Θ) = 0.1009 + 0.0133(Θ).

Between 50 & 70 degrees:
sin(Θ) = 0.3319 + 0.008682(Θ).

Between 70 & 90 degrees:
sin(Θ) = 0.7286 + 0.003015(Θ).

These approximations are accurate to ±0.015 or better, sufficient for our positioning application.

To calculate angles above 90 degrees:

Between 90 & 180 degrees, sin(Θ) = sin(180-Θ).

Between 180 & 270 degrees, sin(Θ) = -sin(Θ-180). Note the minus sign.

Between 270 & 360 degrees, sin(Θ) = -sin(360-Θ). Again, note the minus sign.

We did not do a cosine approximation. Instead, we simply used cos(Θ) = sin(90-Θ). If you want to check all this out, try out a few points on the Figure 2 graphs.

As with the others, if your angle is above 360 or below zero, subtract or add 360 enough times to bring it back to between 0 and 360 degrees

Conclusion.

Computing sine and cosine takes computation time and requires approximations. The “exact” formulas are infinitely long equations. We have presented three approximations, each trading off accuracy for simplicity and speed. Each approximates only one quarter cycle of the sine wave, either -45 to +45 degrees or 0 to 90 degrees. All other angles are related back to that quarter cycle.

Our position control application needed real-time computation but not high accuracy, so we used the simplest of the three.